Kwantlen Polytechnic University
KORA: Kwantlen Open Resource Access

All Faculty Scholarship | Faculty Scholarship

2015

Symmetry Analysis and Conservation Laws for the Class of Time-Fractional Nonlinear Dispersive Equation

Kamran Fakhar
Kwantlen Polytechnic University

Gangwei Wang

A. H. Kara

Follow this and additional works at: http://kora.kpu.ca/facultypub

Part of the Mathematics Commons, and the Non-linear Dynamics Commons

Original Publication Citation

Wang, G., Kara, A.H. & Fakhar, K. (2015). Symmetry analysis and conservation laws for the class of time-fractional nonlinear dispersive equation. Nonlinear Dynamics, May 2015. DOI 10.1007/s11071-015-2156-4

This Article is brought to you for free and open access by the Faculty Scholarship at KORA: Kwantlen Open Resource Access. It has been accepted for inclusion in All Faculty Scholarship by an authorized administrator of KORA: Kwantlen Open Resource Access. For more information, please contact kora@kpu.ca.

Nonlinear Dyn
DOI 10.1007/s11071-015-2156-4

CrossMark

ORIGINAL PAPER

Symmetry analysis and conservation laws for the class of time-fractional nonlinear dispersive equation

Gangwei Wang · A. H. Kara · K. Fakhar

Received: 31 December 2014 / Accepted: 10 May 2015
© Springer Science+Business Media Dordrecht 2015

Abstract In this paper, we derive the complete algebra of Lie point symmetries for the class of ~~time fractional~~time-fractional nonlinear dispersive equation. By means of the classical Lie symmetry method, the associated vector fields are obtained which in turn are utilized for the reduction of the equation. In particular, the conservation laws of the equation are obtained.

Keywords Time-fractional nonlinear dispersive equation · Lie symmetry method · Conservation laws

1 Introduction

Differential equations play an important and central role in many fields. It is well known that Lie theory of symmetry group provides a systemic, general and efficient method to deal with differential equations. This theory is mainly used for the construction of similarity reductions, group invariant solutions and the conservations laws. In general, Lie symmetries can be used to reduce the order as well number of independent variables of original equation (system of equations) . For further details, readers are referred to [1–11] .

Unlike the case of ~~integral order~~integral-orderpartial differential equations (PDEs) , symmetries of ~~fractional order~~fractional-orderpartial differential equations (FPDEs) have not been investigated extensively. The study of FPDEs through symmetries is quite interesting and significant [12–28] . Therefore, in our present ~~study~~study,we investigate the symmetries of FPDEs and thereby do the analysis. We successfully obtain the reduction in independent and dependent variables. Moreover, we look into the key issue of whether we can identify the FPDEs from which the Lie point symmetries are inherited.

In this paper, we will investigate the class of ~~time fractional~~time-fractionalnonlinear dispersive equation

$$u_t^{\alpha} + \varepsilon(u^m)_x + \frac{1}{b}\left[\cancel{u^a(u^b}u^a\left(u^b\right)_{xx}\right]_x = 0, \qquad (1)$$

where $u(x,t)$ represents the wave ~~profile~~profile,while a, b, ε and m are constants. Some special cases of (1) have been used to describe physical situations in various fields. If $\alpha = 1, a = 0$, one can get the generalization of the KdV equation. In the special case if $m = 2, a = 0, b = 1$, Eq. (1) reduces to the classical KdV equation, while $m = 3, a = 0, b = 1$, Eq. (1

G. Wang (✉)
School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, Republic of China
e-mail: pukai1121@163.com; wanggangwei@bit.edu.cn

G. Wang · A. H. Kara · K. Fakhar
Department of Mathematics, University of British Columbia, Vancouver V6T 1Z2, Canada

A. H. Kara
School of Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg Wits 2050, South Africa

K. Fakhar
Department of Mathematics, Faculty of Science and Horticulture, Kwantlen Polytechnic University, Surrey, BC V3W 2MB, Canada

Journal: **11071** MS: **2156** TYPESET DISK LE CP Disp.:**2015/5/15** Pages: **7** Layout: **Medium**

uncorrected proof

) becomes the famous mKdV equation. Further more descriptions of (1) and its applications can be found in [29–32] and references therein.

The paper is divided as follows. In ~~Sec.~~Sect. 2 , some definitions and properties of Lie group method to ~~analysis~~analyzethe FPDEs are given. Moreover, the infinitesimal operators of the ~~Lie-point~~Lie pointsymmetries admitted by Eq. (1) are also constructed. In~~Sec.~~Sect. 3 , the conservation laws of the equation are obtained. The main results of the paper are summarized and discussed in the last section.

2 Lie symmetry analysis of the class of the ~~time fractional~~time-fractionalnonlinear dispersive equation

In this section, we employ Lie symmetry method to deal with the fractional nonlinear dispersive equation. We first briefly recall the concept of fractional derivative ([33–36] and references therein) . In particular, the ~~Riemann-Liouville~~Riemann–Liouvillefractional derivative is defined by

$$D_t^\alpha f(t) = \begin{cases} \cancel{\frac{\partial^n u}{\partial t^n}, \quad \alpha = n,} & \alpha = n \in N, \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(\theta,x)}{(t-\theta)^{\alpha+1-n}} \mathrm{d}\theta, & n-1<\alpha<n, n \in N, \end{cases} \tag{2}$$

where $\Gamma(z)$ is the Euler gamma function.

Assume that (1) is invariant under the one parameter Lie group of point transformations

$$\begin{aligned} t^* &= t + \varepsilon\tau(x,t,u) \cancel{+O(\varepsilon^2} O\left(\varepsilon^2\right), \\ x^* &= x + \varepsilon\xi(x,t,u) \cancel{+O(\varepsilon^2} + O\left(\varepsilon^2\right), \\ u^* &= u + \varepsilon\eta(x,t,u) \cancel{+O(\varepsilon^2} + O\left(\varepsilon^2\right), \\ \frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon \eta_\alpha^0(x,t,u) + O\left(\varepsilon^2\right), \\ \frac{\partial \bar{u}}{\partial \bar{x}} &= \frac{\partial u}{\partial x} + \varepsilon \eta^x(x,t,u) \cancel{+O(\varepsilon^2} + O\left(\varepsilon^2\right), \\ \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} &= \frac{\partial^2 u}{\partial x^2} + \varepsilon \eta^{xx}(x,t,u) \cancel{+O(\varepsilon^2} + O\left(\varepsilon^2\right), \\ \frac{\partial^3 \bar{u}}{\partial \bar{x}^3} &= \frac{\partial^3 u}{\partial x^3} + \varepsilon \eta^{xxx}(x,t,u) \cancel{+O(\varepsilon^2),} + O\left(\varepsilon^2\right), \end{aligned} \tag{3}$$

where ε is the group parameter, and its associated Lie algebra is spanned by the following vector fields

$$V = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{4}$$

Here,

$$\tau(x,t,u) = \frac{\mathrm{d}t^*}{\mathrm{d}\varepsilon}|_{\varepsilon=0}, \ \xi(x,t,u) = \frac{\mathrm{d}x^*}{\mathrm{d}\varepsilon}|_{\varepsilon=0},$$
$$\eta(x,t,u) = \frac{\mathrm{d}u^*}{\mathrm{d}\varepsilon}|_{\varepsilon=0}. \tag{5}$$

On the basis of the infinitesimal invariance criterion, one can get

$$pr^{(\alpha,3)} V(\Delta_1)|_{\Delta_1=0} = 0, \tag{6}$$

where $\Delta_1 = u_t^\alpha + \left(\varepsilon(u^m)_x + \frac{1}{b}\left[\cancel{u^a(u^b} u^a\left(u^b\right)_{xx}\right]_x\right)$.

The prolongation operator $pr^{(\alpha,3)}V$ is

$$\begin{aligned} pr^{(\alpha,3)}V &= V + \eta_\alpha^0 \partial_{\partial_t^\alpha u} + \eta^x \partial_{u_x} + \eta^{xx}\partial_{u_{xx}} \\ &\quad + \eta^{xxx}\partial_{u_{xxx}}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \eta^x &= \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t, \\ \eta^{xx} &= \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t \\ &\quad + (\eta_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_t - \xi_{uu}\cancel{u_x^3} - u_x^3 \\ &\quad - \tau_{uu}u_x^2 u_t + (\eta_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} \\ &\quad - 3\xi_u u_{xx}u_x - \tau_u u_{xx}u_t - 2\tau_u u_{xt}u_x, \\ \eta^{xxx} &= \eta_{xxx} + (3\eta_{xxu} - \xi_{xxx})u_x - \tau_{xxx}\cancel{u_t + 3}u_t \\ &\quad + 3(\eta_{xuu-\xi_{xxu}})u_x^2 - 3\tau_{xxu}u_x u_t + u_x u_t \\ &\quad + (\eta_{uuu} - 3\xi_{xuu})\, u_x^3 + 3(\eta_{xu} - \xi_{xx})u_{xx} \\ &\quad - 3\tau_{xx}u_{xt} - 3\tau_{xuu}u_x^2 u_t + 3(\eta_{uu} - 3\xi_{xu})u_x u_{xx} \\ &\quad - 3\tau_{xu}u_{xx}u_t - 6\tau_{xu}u_{xt}u_x - 3\tau_x u_{xxt}\cancel{+} \\ &\quad + (\eta_u - 3\xi_x)u_{xxx} - \xi_{xxx}u_x^4 \\ &\quad - 6\xi_{uu}u_{xx}u_x^2 - 3\tau_{uu}u_x^2 u_{tx} - \tau_{uuu}\cancel{u_x^3 u_t - 3}u_x^3 u_t \\ &\quad - 3\xi_u u_{xx}^2 - 3\tau_u u_{xxt}u_x - 3\tau_u u_{xt}u_{xx}\cancel{-3} \\ &\quad - 3\tau_{uu}u_t u_x u_{xx} - 4\xi_u u_x u_{xxx} - \tau_u u_t u_{xxx}. \end{aligned} \tag{8}$$

In particular,

$$\begin{aligned} \eta_\alpha^0 &= \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} \\ &\quad + \mu + \sum_{n=1}^{\infty}\left[\binom{a}{n}\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \binom{a}{n+1}D_t^{n+1}(\tau)\right] \\ &\quad \times D_t^{\alpha-n}(u) - \sum_{n=1}^{\infty}\binom{a}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x), \end{aligned} \tag{9}$$

where

$$\begin{aligned} \cancel{\mu =}\mu &= \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{a}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \\ &\quad \times [-u]^r \frac{\partial^m}{\partial t^m}[u^{k-r}]\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}, \end{aligned} \tag{10}$$

Journal: 11071 MS: 2156 TYPESET DISK LE CP Disp.:2015/5/15 Pages: 7 Layout: Medium

Uncorrected proof

and additional constraint condition is

$$\tau(x,t,u)|_{t=0}=0. \tag{11}$$

Compared with the Lie symmetry method to ~~integral order~~integral-orderdifferential equations, it is can be easily seen that constraint condition (11) and formula (9) are critical to FPDEs.

Now, we will study the class of ~~time fractional~~time-fractionalnonlinear dispersive equation using above Lie symmetry group theory. First, one can get the following assertion

Theorem 1 *The symmetry group of the equation is spanned by the following vector fields*

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x},\ V_2=\frac{(a+b-3m+2)\,t}{\alpha}\frac{\partial}{\partial t}\\ &\quad +((a+b-m)x)\frac{\partial}{\partial x}+2u\frac{\partial}{\partial u}. \end{aligned}\tag{12}$$

Proof By assuming that Eq. (1) is invariant under the transformation group (3) , one can get the symmetry equation as follows

$$\begin{aligned} &\eta_\alpha^0+(a+b-1)\eta u^{a+b-2}u_{xxx}\\ &+\eta^{xxx}u^{a+b-1}+\varepsilon m(m-1)\eta u^{m-2}u_x\\ &+\varepsilon m\eta^x u^{m-1}+(a+3b-3)(a+b-2)\eta u^{a+b-3}u_xu_{xx}\\ &+(a+3b-3)\eta^x u^{a+b-2}u_{xx}\\ &+(a+3b-3)\eta^{xx}u^{a+b-2}u_x\\ &+(b-1)(a+b-2)(a+b-3)\eta u^{a+b-4}u_x^3\\ &+3(b-1)(a+b-2)\eta^x u^{a+b-3}u_x^2=0. \end{aligned}\tag{13}$$

Substituting (8) ~~/~~–(11) into (13) , and letting all of the powers of derivatives of u to zero, one can have

$$\begin{aligned} &\xi_u=\tau_u=\xi_t=\xi_{xx}=\tau_x=\eta_{uu}=0,\\ &(\tau_t\alpha-3\xi_x)u+(a+b-1)\eta=0,\\ &(\tau_t\alpha+\eta_u-3\xi_x)u+(a+b-2)\eta=0,\\ &(\tau_t\alpha+2\eta_u-3\xi_x)u+(a+b-3)\eta=0,\\ &(\tau_t\alpha-\xi_x)u+(m-1)\eta=0,\\ &\binom{a}{n}\partial_t^n(\eta_u)-\binom{a}{n+1}D_t^{n+1}(\tau)=0,\\ &\text{for } n=1,2,\cancel{,}\ldots \end{aligned}\tag{14}$$

By solving these equations, we have

$$\begin{aligned} &\xi=c_1(a+b-m)x+c_2,\quad \tau=\frac{c_1\,(a+b-3m+2)\,t}{\alpha},\\ &\eta=2c_1u, \end{aligned}\tag{15}$$

here c_1 and c_2 are arbitrary constants. Thus, the corresponding vector fields are

$$\begin{aligned} V&=\frac{c_1\,(a+b-3m+2)\,t}{\alpha}\frac{\partial}{\partial t}\\ &\quad+\left[c_1\,(a+b-m)\,x+c_2\right]\frac{\partial}{\partial x}+2c_1u\frac{\partial}{\partial u}. \end{aligned}\tag{16}$$

or

$$\begin{aligned} V_1&=\frac{\partial}{\partial x},\ V_2=\frac{(a+b-3m+2)\,t}{\alpha}\frac{\partial}{\partial t}\\ &\quad+\left((a+b-m)x\right)\frac{\partial}{\partial x}+2u\frac{\partial}{\partial u}, \end{aligned}\tag{17}$$

which complete the proof. □

In particular, for the symmetry V_2, we have the characteristic equation

$$\frac{dx}{(a+b-m)\,x}=\frac{\alpha \mathrm{d}t}{(a+b-3m+2)\,t}=\frac{\mathrm{d}u}{2u},\tag{18}$$

which leads to the following similarity variable and the similarity transformation

$$\xi=xt^{-\frac{\alpha(a+b-m)}{a+b-3m+2}},\quad u=t^{\frac{2\alpha}{a+b-3m+2}}g(\xi),\tag{19}$$

as ~~required~~required.

Theorem 2 *The transformation (19) reduces (1) to the following nonlinear ODE of* ~~*fractional order*~~*fractional-order*

$$\begin{aligned} &\left(P^{1-\alpha+\frac{2\alpha}{a+b-3m+2},\alpha}_{\frac{a+b-3m+2}{\alpha(a+b-m)}}\right)g(\xi)\\ &+\varepsilon mg^{m-1}g_\xi+(b-1)(a+b-2)g^{a+b-3}g_\xi^3\\ &+(a+b-3)g^{a+b-2}g_{\xi\xi}g_\xi+g^{a+b-1}g_{\xi\xi\xi}=0, \end{aligned}\tag{20}$$

with the ~~*Erdelyi-Kober*~~*Erdelyi–Koberfractional differential operator* $P_\beta^{\tau,\alpha}$ *of order* [33]

$$(P_\beta^{\tau,\alpha}g):=\prod_{j=0}^{n-1}\left(\tau+j-\frac{1}{\beta}\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\left(K_\beta^{\tau+\alpha,n-\alpha}\cancel{g}g\right)(\xi),\tag{21}$$

$$n=\begin{cases}[\alpha]+1, & \alpha\notin N,\\ \alpha & \alpha\in N,\end{cases}\tag{22}$$

where

$$\begin{aligned} &\left(K_\beta^{\tau,\alpha}\cancel{g}g\right)(\xi)\\ &:=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_1^\infty(u-1)^{\alpha-1}u^{-(\tau+\alpha)}g(\xi u^{\frac{1}{\beta}})\mathrm{d}u, & \alpha>0,\\ g(\xi), & \alpha=0,\end{cases} \end{aligned}\tag{23}$$

is the Erdé lyi–Kober fractional integral operator.

uncorrected proof

Proof We first let $n-1<\alpha<n$, $n=1,2,3,\ldots$. Then, in light of the ~~Riemann-Liouville~~Riemann–Liouvillefractional derivative and the similarity transformation, we get

$$\frac{\partial^\alpha u}{\partial t^\alpha} \cancel{=} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} \times s^{\frac{2\alpha}{a+b-3m+2}} g\left(xs^{-\frac{\alpha(a+b-m)}{a+b-3m+2}}\right)\mathrm{d}s\right]. \tag{24}$$

Under the assumption $v=\frac{t}{s}$, Eq. (24) reduces

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial^n}{\partial t^n}\left[t^{n-\alpha+\frac{2\alpha}{a+b-3m+2}}\frac{1}{\Gamma(n-\alpha)}\int_1^\infty (v-1)^{n-\alpha-1}\right.\\ &\qquad \left.\times v^{-\left(n-\alpha+\frac{2\alpha}{a+b-3m+2}+1\right)} g\left(\xi v^{\frac{\alpha(a+b-m)}{a+b-3m+2}}\right)\mathrm{d}v\right]\\ &= \frac{\partial^n}{\partial t^n}\left[t^{n-\alpha+\frac{2\alpha}{a+b-3m+2}}\left(K^{1+\frac{2\alpha}{a+b-3m+2},\,n-\alpha}_{\frac{a+b-3m+2}{\alpha(a+b-m)}} g\right)(\xi)\right].\end{aligned} \tag{25}$$

By using the relation $\xi = xt^{-\frac{\alpha(a+b-m)}{a+b-3m+2}}$, Eq. (25) further simplifies to

$$\begin{aligned}t\frac{\partial}{\partial t}\phi(\xi) &= tx\left(-\frac{\alpha(a+b-m)}{a+b-3m+2}\right)t^{-\frac{\alpha(a+b-m)}{a+b-3m+2}-1}\phi'(\xi)\\ &= -\frac{\alpha(a+b-m)}{a+b-3m+2}\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi).\end{aligned} \tag{26}$$

Thus, one can get

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha+\frac{2\alpha}{a+b-3m+2}}\left(K^{1+\frac{2\alpha}{a+b-3m+2},\,n-\alpha}_{\frac{a+b-3m+2}{\alpha(a+b-m)}} g\right)(\xi)\right]\\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\alpha+\frac{2\alpha}{a+b-3m+2}}\right.\right.\\ &\quad \left.\left.\times\left(K^{1+\frac{2\alpha}{a+b-3m+2},\,n-\alpha}_{\frac{a+b-3m+2}{\alpha(a+b-m)}} \cancel{g}g\right)(\xi)\right)\right]\\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\alpha+\frac{2\alpha}{a+b-3m+2}-1}\left(n-\alpha\right.\right.\\ &\quad \left.+\frac{2\alpha}{a+b-3m+2}-\frac{\alpha(a+b-m)}{a+b-3m+2}\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\\ &\quad \left.\times\left(K^{1+\frac{2\alpha}{a+b-3m+2},\,n-\alpha}_{\frac{a+b-3m+2}{\alpha(a+b-m)}} g\right)(\xi)\right].\end{aligned} \tag{27}$$

~~Repeat~~Repeatingthe previous step, one has

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha+\frac{2\alpha}{a+b-3m+2}}\left(K^{1+\frac{2\alpha}{a+b-3m+2},\,n-\alpha}_{\frac{a+b-3m+2}{\alpha(a+b-m)}} g\right)(\xi)\right]\\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\alpha+\frac{2\alpha}{a+b-3m+2}}\right.\right.\\ &\quad \left.\left.\times\left(K^{1+\frac{2\alpha}{a+b-3m+2},\,n-\alpha}_{\frac{a+b-3m+2}{\alpha(a+b-m)}} \cancel{g}g\right)(\xi)\right)\right]\\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\alpha+\frac{2\alpha}{a+b-3m+2}-1}\left(n-\alpha\right.\right.\\ &\quad \left.+\frac{2\alpha}{a+b-3m+2}-\frac{\alpha(a+b-m)}{a+b-3m+2}\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\\ &\quad \left.\times\left(K^{1+\frac{2\alpha}{a+b-3m+2},\,n-\alpha}_{\frac{a+b-3m+2}{\alpha(a+b-m)}} g\right)(\xi)\right]\\ &= \cdots = t^{n-\alpha+\frac{2\alpha}{a+b-3m+2}}\prod_{j=0}^{n-1}\left(1-\alpha\right.\\ &\quad \left.+\frac{2\alpha}{a+b-3m+2}+j-\frac{\alpha(a+b-m)}{a+b-3m+2}\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\\ &\quad \times\left(K^{1+\frac{2\alpha}{a+b-3m+2},\,n-\alpha}_{\frac{a+b-3m+2}{\alpha(a+b-m)}} g\right)(\xi).\end{aligned} \tag{28}$$

So we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\alpha+\frac{2\alpha}{a+b-3m+2}}\left(P^{1-\alpha+\frac{2\alpha}{a+b-3m+2},\,\alpha}_{\frac{a+b-3m+2}{\alpha(a+b-m)}} g\right)(\xi). \tag{29}$$

Hence, it is easily found that (1) reduces into the following ~~fractional order~~fractional-orderODE

$$\begin{aligned}&\left(P^{1-\alpha+\frac{2\alpha}{a+b-3m+2},\,\alpha}_{\frac{a+b-3m+2}{\alpha(a+b-m)}}\right) g(\xi) + \varepsilon m g^{m-1} g_\xi\\ &\quad +(b-1)(a+b-2)g^{a+b-3}g_\xi^3\\ &\quad +(a+b-3)g^{a+b-2}g_{\xi\xi}g_\xi + g^{a+b-1}g_{\xi\xi\xi}=0.\end{aligned} \tag{30}$$

This completes the proof. □

3 Conservation laws

In this section, we study the conservation laws of the class of time-fractional nonlinear dispersive equation. The ~~Riemann-Liouville~~Riemann–Liouvilleleft-sided time-fractional derivative will be used as

$${}_0D_t^\alpha u = D_t^n\left({}_0I_t^{n-\alpha}u\right), \tag{31}$$

in Eq. (1) . ~~Here~~Here, D_t is the operator of differentiation with respect to t, $n=[\alpha]+1$, and ${}_0I_t^{n-\alpha}u$ is the left-sided time-fractional integral of order $n-\alpha$ defined by [25]

$$\left({}_0I_t^{n-\alpha}u\right)(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u(\theta,x)}{(t-\theta)^{1-n+\alpha}}\mathrm{d}\theta, \tag{32}$$

where $\Gamma(z)$ is the Gamma function.

uncorrected proof

Journal: **11071** MS: **2156** ☐ TYPESET ☐ DISK ☐ LE ☐ CP Disp.:**2015/5/15** Pages: **7** Layout: **Medium**

3.1 Necessary preliminaries

A conserved vector satisfies the following conservation equation

$$D_t(C^t) + D_x(C^x) = 0, \tag{33}$$

where $C^t = C^t(t, x, u, \cancel{/}\ldots)$, $C^x = C^x(t, x, \cancel{u, \ldots} u, \ldots)$. Eq. (33) is called a conservation law for Eq. (1) .

A formal Lagrangian for (1) can be introduced as

$$L = v(x,t)\left[u_t^{\alpha} + \varepsilon\left(u^m\right)_x + \frac{1}{b}\left[\cancel{u^a(u^b}u^a\left(u^b\right)_{xx}\right]_x\right] \tag{34}$$

~~Here~~Here, $v(x, t)$ is a new dependent variable. Considering the formal Lagrangian, an action integral is given by

$$\int_0^T \int_{\Omega} L\left(x, t, u, v, D_t^{\alpha}u, u_x \cdots\right) \mathrm{d}x\mathrm{d}t. \tag{35}$$

The ~~Euler-Lagrange~~Euler–Lagrange operator is defined by [25,26]

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^{\alpha})^* \frac{\partial}{\partial D_t^{\alpha}u} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} - D_x^3\frac{\partial}{\partial u_{xxx}}, \tag{36}$$

where $(D_t^{\alpha})^*$ is the adjoint operator of (D_t^{α}).

Note that Eq. (1) with the ~~Riemann-Liouville~~Riemann–Liouvillefractional derivative can be rewritten in the form of conservation law form (33) with

$$C^t = D_t^{n-1}({}_0I_t^{n-\alpha}u),\ C^x = \varepsilon u^m + \frac{1}{b}u^a(u^b)_{xx}. \tag{37}$$

The adjoint equation is similarly to the case of integer-order nonlinear differential equations [25,26] , so we have the adjoint equation to the nonlinear TFDE (1) as Euler–Lagrange equation

$$\frac{\delta L}{\delta u} = 0. \tag{38}$$

Considered the case of two independent variables t, x, and one dependent variable $u(t, x)$, this fundamental identity can be written as

$$\bar{X} + D_t(\tau)l + D_x(\xi)l = W\frac{\delta}{\delta u} + D_tN^t + D_xN^x, \tag{39}$$

where l is the identity operator, $\frac{\delta}{\delta u}$ is the ~~Euler-Lagrange~~Euler–Lagrangeoperator, N^t and N^x are the Noether operators, $\bar{X}$ is an appropriate prolongation for the Lie point generator

$$\bar{X} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \eta_{\alpha}^0\frac{\partial}{\partial D_t^{\alpha}u} + \eta^x\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xxx}\frac{\partial}{\partial u_{xxx}}, \tag{40}$$

and

$$W = \eta - \tau u_t - \xi u_x. \tag{41}$$

For the ~~case Riemann-Liouville~~case, Riemann–Liouvilletime-fractional derivative is used in Eq. (1) , the operator N^t is given by [25,26]

$$N^t = \tau l + \sum_{k=0}^{n-1}(-1)^k\,{}_0D_t^{\alpha-1-k}(W)D_t^k\frac{\partial}{\partial_0 D_t^{\alpha}u} - (-1)^n J\left(W, D_t^n\frac{\partial}{\partial_0 D_t^{\alpha}u}\right), \tag{42}$$

where J is the integral [25,26]

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T \frac{f(\tau,x)g(\mu,x)}{(\mu-\tau)^{\alpha+1-n}}\mathrm{d}\mu\mathrm{d}t. \tag{43}$$

The operator N^x is defined by

$$N^x = \xi l + W\left(\frac{\partial}{\partial u_x} - D_x\frac{\partial}{\partial u_{xx}} + D_x^2\frac{\partial}{\partial u_{xxx}}\right) + D_x(W)\left(\frac{\partial}{\partial u_{xx}} - D_x\frac{\partial}{\partial u_{xxx}}\right)\cancel{+D_x^2} + D_x^2(W)\frac{\partial}{\partial u_{xxx}}. \tag{44}$$

For any generator X admitted by Eq. (1) and any solution of this equation, we have:

$$\left(\bar{X}L + D_t(\tau)L + D_x(\xi)L\right)|_{(1)} = 0. \tag{45}$$

This equality yields the conservation law

$$D_t(N^tL) + D_x(N^xL) = 0. \tag{46}$$

3.2 Conservation laws

In the previous subsection, we gave some basic definitions. In this subsection, we will present the conservation laws.

uncorrected proof

Journal: 11071 MS: 2156 TYPESET DISK LE CP Disp.:2015/5/15 Pages: 7 Layout: Medium

For the case, when $\alpha \in (0, 1)$, using (42) and (44) , one can get the components of conserved vectors

$$C_i^t = \tau L + (-1)^0 \, {}_0D_t^{\alpha-1}(W_i) D_t^0 \frac{\partial L}{\partial \left\{ {}_0D_t^{\alpha} u \right\}} - (-1)^1 J \left(W_i, D_t^1 \frac{\partial L}{\partial \left\{ {}_0D_t^{\alpha} u \right\}} \right)$$
$$= v_0 D_t^{\alpha-1}(W_i) + J(W_i, v_t), \tag{47}$$

$$C_i^x = \xi L + W_i \left(\frac{\partial L}{\partial u_x} - D_x \frac{\partial L}{\partial u_{xx}} + D_x^2 \frac{\partial L}{\partial u_{xxx}} \right) + D_x(W_i) \left(\frac{\partial L}{\partial u_{xx}} - D_x \frac{\partial L}{\partial u_{xxx}} \right) + D_x^2(W_i) \frac{\partial L}{\partial u_{xxx}}$$
$$= W_i \Big\{ v \Big((a+b-3) u^{a+b-2} u_{xx} + 3(b-1)(a+b-2) u^{a+b-3} u_x^2 \cancel{+\varepsilon} + \varepsilon m u^{m-1} \Big) - D_x \Big[v(a+b-3) u^{a+b-2} u_x \Big] + D_x^2 \left(v u^{a+b-1} \right) \Big\}$$
$$+ D_x(W_i) \Big\{ v(a+3b-3) u^{a+b-2} \cancel{u_x - D_x} u_x - D_x \left(v u^{a+b-1} \right) \Big\} + D_x^2(W_i) \left(v u^{a+b-1} \right), \tag{48}$$

where $i = 1, 2$ and functions W_i are

$$W_1 = -u_x, \ W_2 = 2u - \frac{(a+b-3m+2)\, t}{\alpha} u_t - \big[(a+b-m)\, x \big] u_x. \tag{49}$$

Also, when $\alpha \in (1, 2)$, we get the components of conserved vectors

$$C_i^t = \tau L + (-1)^0 \, {}_0D_t^{\alpha-1}(W_i) D_t^0 \frac{\partial L}{\partial \left\{ {}_0D_t^{\alpha} u \right\}} - (-1)^1 J \left(W_i, D_t^1 \frac{\partial L}{\partial \left\{ {}_0D_t^{\alpha} u \right\}} \right)$$
$$+ (-1)^1 {}_0D_t^{\alpha-2}(W_i) D_t^1 \frac{\partial L}{\partial \left\{ {}_0D_t^{\alpha} u \right\}} - (-1)^2 J \left(W_i, D_t^2 \frac{\partial L}{\partial \left\{ {}_0D_t^{\alpha} u \right\}} \right)$$
$$= v \, {}_0D_t^{\alpha-1}(W_i) + J(W_i, v_t) - v_t \, {}_0D_t^{\alpha-2}(W_i) - J(W_i, v_{tt}), \tag{50}$$

$$C_i^x = \xi L + W_i \left(\frac{\partial L}{\partial u_x} - D_x \frac{\partial L}{\partial u_{xx}} + D_x^2 \frac{\partial L}{\partial u_{xxx}} \right) + D_x(W_i) \left(\frac{\partial L}{\partial u_{xx}} - D_x \frac{\partial L}{\partial u_{xxx}} \right) + D_x^2(W_i) \frac{\partial L}{\partial u_{xxx}}$$
$$= W_i \Big\{ v \Big((a+b-3) u^{a+b-2} u_{xx} + 3(b-1)(a+b-2) u^{a+b-3} u_x^2 + \varepsilon m u^{m-1} \Big) - D_x \Big[v(a+b-3) u^{a+b-2} u_x \Big] + D_x^2 \left(v u^{a+b-1} \right) \Big\}$$
$$+ D_x(W_i) \Big\{ v(a+3b-3) u^{a+b-2} \cancel{u_x - D_x} u_x - D_x \left(v u^{a+b-1} \right) \Big\} + D_x^2(W_i) \left(v u^{a+b-1} \right), \tag{51}$$

where $i = 1, 2$ and functions W_i have the form

$$W_1 = -u_x, \ W_2 = 2u - \frac{(a+b-3m+2)\, t}{\alpha} u_t - \big[(a+b-m)\, x \big] u_x. \tag{52}$$

4 Concluding remarks and discussion

In this paper, we investigated ~~time fractional~~time-fractionalnonlinear dispersive equation via Lie symmetries and conservation laws. We firstly obtained the Lie point symmetries and perform symmetry reductions. Furthermore, the conservation laws are constructed for the first time in this paper. The obtained results will serve as benchmark in the accuracy testing, comparison of numerical results. There are several issues which need to be pursued furthers. For example, here we have used classical Lie symmetry method for only two independent x, t and one dependent u variables. It is not clear that, how to derive similar results in the case of time FPDEs with more independent and dependent variables. In addition, for the conservative form $(u)_t^{\alpha} + \left(\varepsilon u^m + \frac{1}{b} \cancel{u^a (u^b} u^a \left(u^b\right)_{xx} \right)_x = 0$, one can write the potential system $u = v_x$ and $\varepsilon u^m + \frac{1}{b} \cancel{u^a (u^b} u^a \left(u^b\right)_{xx} = -v_t^{\alpha}$, it is also not clear that whether there exists nonlocal symmetry. It is of interest in general to study whether the method of investigating PDEs can be extended to FPDEs. It is worthy of investigating further and these topics will be reported in the future series of research works.

Acknowledgments The authors are thankful for the referees useful suggestions which help us to improve the manuscript. This work is supported by the Research Project of China Scholarship Council (No. 201406030057) , National Natural Science Foundation of China (NNSFC) (Grant No. 11171022) , Graduate Student Science and Technology Innovation Activities of Beijing Institute of Technology (No. 2014cx10037) .

uncorrected proof

References

1. Bluman, G.W., Kumei, S.: Symmetries and Differential Equations. Springer, New York (1989)
2. Ibragimov, N.H. (ed.) : CRC Handbook of Lie Group Analysis of Differential Equations, vol. 1–3. CRC Press, Boca ~~Raton, FL~~Raton (1994)
3. Bluman, G.W., Cheviakov, A., Anco, S.: Applications of Symmetry Methods to Partial Differential Equations. Springer, New York (2010)
4. Bluman, G.W., Anco, S.: Symmetry and Integration Methods for Differential Equations. Springer, New York (2002)
5. Olver, P.J.: Application of Lie Group to Differential Equation. Springer, New York (1986)
6. Ovsiannikov, L.V.: Group Analysis of Differential Equations. Academic Press, New York (1982)
7. Gang, G.W., et al.: Singular solitons, shock waves, and other solutions to potential KdV equation. Nonlinear Dyn. **76** , 1059–1068 (2014)
8. Biswas, A., et al.: Solitons and conservation laws of ~~Klein-Gordon~~Klein–Gordonequation with power law and log law nonlinearities. Nonlinear Dyn. **73** , 2191–2196 (2013)
9. Biswas, A.: Solitary waves for power-law regularized long-wave equation and R (m, n) equation. Nonlinear Dyn. **59** , 423–426 (2010)
10. Razborova, P., Kara, A.H., Biswas, A.: Additional conservation laws for ~~RosenauCKdVCRLW~~Rosenau–KdV–RLWequation with power law nonlinearity by Lie symmetry. Nonlinear Dyn. **79** , 743–748 (2015)
11. Gazizov, R.K., Kasatkin, A.A.: Construction of exact solutions for fractional order differential equations by the invariant subspace method. Comput. Math. Appl. **66** , 576–584 (2013)
12. Liu, H.Z.: Complete group classifications and symmetry reductions of the fractional fifth-order KdV types of equations. Stud. Appl. Math. **131** , 317–330 (2013)
13. Gazizov, ~~R. K.,~~R.K.,Kasatkin, ~~A. A.,~~A.A.,Yu, S.: Lukashchuk, Continuous transformation groups of fractional differential equations, Vestnik, USATU 9 125–135 (in Russian) (2007)
14. Buckwar, E., Luchko, Y.: Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations. J. Math. Anal. Appl. **227** , 81–97 (1998)
15. Djordjevic, V.D., Atanackovic, T.M.: Similarity solutions to nonlinear heat conduction and Burgers/~~Korteweg-deVries~~Burgers/Korteweg–deVriesfractional equations. J. Comput. Appl. Math. **212** , 701–714 (2008)
16. Gazizov, R.K., Kasatkin, A.A., Lukashchuk, SYu.: Symmetry properties of fractional diffusion equations. Phys. Scr. **T136** , 014016 (2009)
17. Sahadevan, R., Bakkyaraj, T.: Invariant analysis of time fractional generalized Burgers and ~~Korteweg-de~~Korteweg–deVries equations. J. Math. Anal. Appl. **393** , 341–347 (2012)
18. Wang, G.W., Liu, X.Q., Zhang, Y.Y.: Lie symmetry analysis to the time fractional generalized fifth-order KdV equation. Commun. Nonlinear Sci. Numer. ~~Simulat.~~Simul.**18** , 2321–2326 (2013)
19. Wang, G.W., Xu, T.Z., Feng, T.: Lie symmetry analysis and explicit solutions of the time fractional fifth-Order KdV equation. Plos One **9** (2) , e88336 (2014)
20. Wang, G.W., Xu, T.Z.: Invariant analysis and exact solutions of nonlinear time fractional ~~Sharma-Tasso-Olver~~Sharma–Tasso–Olverequation by Lie group analysis. Nonlinear Dyn. **76** , 571–580 (2014)
21. Wang, ~~G. W.,~~G.W.,Xu, ~~T. Z.:~~T.Z.:Symmetry properties and explicit solutions of the nonlinear time fractional KdV equation, Bound. Value Probl. (2013) 4
22. Huang, Q., Zhdanov, R.: Symmetries and exact solutions of the time fractional ~~Harry-Dym~~Harry–Dymequation with Riemann-Liouville derivative. ~~Physica~~Phys.A **409** , 110–118 (2014)
23. Hu, J., Ye, Y., Shen, S., Zhang, J.: Lie symmetry analysis of the time fractional KdV-type equation. Appl. Math. Comput. **233** , 439–444 (2014)
24. Hashemi, M.S.: Group analysis and exact solutions of the time fractional ~~Fokker-Planck~~Fokker–Planckequation. ~~Physica~~Phys.A **417** , 141–149 (2015)
25. Lukashchuk, S.Y.: Conservation laws for time-fractional subdiffusion and diffusion-wave equations. Nonlinear Dyn. (2015) . doi:10.1007/s11071-015-1906-7
26. Gazizov, R.K., Ibragimov, N.H., Lukashchuk, S.Y.: Nonlinear self-adjointness, conservation laws and exact solutions of time-fractional Kompaneets equations. Commun. Nonlinear Sci. Numer. ~~Simulat.~~Simul. (2014) . doi:10.1016/j.cnsns.2014.11.010
27. Wang, ~~G. W.,~~G.W.,Hashemi, ~~M.S. :~~M.S.:Lie symmetry analysis and solitons solutions of time fractional K (m, n) ~~equation,~~equation (submitteds*)s*
28. Wang, ~~G. W.,~~G.W.,Kara, ~~A. H. :~~A.H.:Group analysis and soliton solution of the time fractional generalized Burgers ~~equation,~~equation (submitteds*)s*
29. Rosenau, P.: On nonalnalytic solitary waves formed by a nonlinear dispersion. Phys. Lett. A **230** , 305–318 (1997)
30. Rosenau, P.: Compact and noncompact dispersive patterns. Phys. Lett. A **275** , 193–203 (2000)
31. Rosenau, P.: On a model equation of traveling and stationary compactons. Phys. Lett. A **356** , 44–50 (2006)
32. Charalambous, K., Sophocleous, C.: Symmetry analysis for a class of nonlinear dispersive equations. Commun. Nonlinear Sci. Numer. ~~Simulat.~~Simul.**22** , 1275–1287 (2015)
33. Kiryakova, V. : Generalised Fractional Calculus and Applications. In: Pitman Res. Notes in Math., vol. 301, (1994)
34. Liu, C.S.: Counterexamples on Jumarie's two basic fractional calculus formulae. Commun. Nonlinear Sci. Numer. ~~Simulat.~~Simul.**22** , 92–94 (2015)
35. Mainardi, F.: Fractional Calculus and Waves in Linear Viscoelasticity. World Scientific, Singapore (2010)
36. Podlubny, ~~I. :~~I.:Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their ~~applications,~~applications.Academic press, New York (1998)

uncorrected proof